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Schauder-Type Expansions of Continuous Functions on the Unit Interval

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Let the space of continuous functions on [0, 1] which vanish at 0 be denoted by C. It will be shown that for any complete orthonormal set of functions $\{\alpha_i(s)\}$ of bounded variation and such that $\alpha_i(1) = 0$, there is a simply described linear combination of the continuous functions $\{\int_0^t \alpha_i(s)\, ds\}$ which converges uniformly to $x(t)$ for almost all $x \in C$ ("almost all" in the sense of Wiener measure).

Let

$$\beta_i(t) = \int_0^t \alpha_i(s)\, ds,$$

$$c_i(x) = \int_0^1 \alpha_i(s)\, dx(s) \qquad \left(= -\int_0^1 x(s)\, d\alpha_i(s)\right).$$

In [3, p. 29] four motivations are given for the question of the representation of $x(t)$, in some sense as

$$x(t) \sim \sum_{i=1}^{\infty} c_i(x)\, \beta_i(t).$$

Here it will be noted only that this representation arises naturally in an intuitive way if one takes the indefinite integral from 0 to t of the partial sums of the $\{\alpha_i(s)\}$ expansion of $x'(s)$. The theorem below follows easily from a result of Gross.

THEOREM. *Let $\{\alpha_i(s)\}$ be as above. There exists a subsequence $\{n_j\}$ of the positive integers such that*

$$\sum_{i=1}^{n_j} c_i(x)\, \beta_i(t)$$

converges uniformly to $x(t)$ on $[0, 1]$ for almost all $x \in C$.

Thus any β sequence is somewhat like a Schauder basis for C. In fact, Ciesielski [1, 1.4] has shown that if the α's are the Haar functions the β's are a Schauder basis. ([2, p. 67] outlines a proof of this.)

Before a proof of the theorem is given, two observations about possible applications will be made. The first is that, as noted in [3, p. 37], the $c\beta$ series sometimes is partly an orthonormal series and possibly the information obtained from the theorem for the latter (for almost all x) can be useful. For example, if the α's are chosen to be the cosine functions, the $c\beta$ series is

$$\sum_{k=1}^{n} \int_0^1 x(s)(2)^{1/2} \sin k\pi s \, ds \, (2)^{1/2} \sin k\pi t + x(1) \left[t - 2 \sum_{k=1}^{n} (-1)^k \sin k\pi t / k\pi \right].$$

From the known behavior of the sine expansion of t it follows at once that a subsequence of the sine expansion of $x(t)$ converges uniformly to $x(t)$ (on $[0, 1 - \epsilon]$ with Gibbs phenomenon at 1) for almost all x. The uniform convergence part of the result is of course not new, since an even stronger result (convergence of the entire sequence) is implied by a result in [7, p. 537] and a theorem of Wiener [8] that almost all x's satisfy a Hölder condition on $[0, 1]$. This example of the choice of cosines for the α's is also found in [6, (2), p. 23; 9, (1), p. 330, and Remark 23.5, p. 338].

The second observation is that if $F[x]$ is bounded and continuous in the uniform topology on C, then the Wiener integral

$$\int_C F[x] \, dx$$

can be found by replacing x by its $c\beta$ nth partial sum and taking the limit as $n \to \infty$. This fact is an improvement on a part of Theorem 2 [2, p. 64].

The proof of the theorem will now be given. Gross' Corollary 5.2 [4, p. 386] (also [5, p. 174, footnote]) for classical Wiener space says that for any two α-sets $\{\alpha_i\}$ and $\{\alpha_i^*\}$,

$$\sup_{t \in [0,1]} \left| \sum_{i=1}^{n} c_i(x) \, \beta_i(t) - \sum_{i=1}^{n} c_i^*(x) \, \beta_i^*(t) \right|$$

converges to zero in measure as $n \to \infty$. This in turn implies that for some subsequence $\{n_j\}$

$$\sup_{t \in [0,1]} \left| \sum_{i=1}^{n_j} c_i(x) \, \beta_i(t) - \sum_{i=1}^{n_j} c_i^*(x) \, \beta_i^*(t) \right|$$

converges to zero for almost all x. As noted above, Ciesieslki's result shows that if the α^*'s are the Haar functions the $c^*\beta^*$ sequence converges uniformly to $x(t)$. The conclusion of the theorem follows at once.

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